ADO-IWASAWA EXTRAS

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(Received 22 November 2003; revised 17 March 2004)

Communicated by E. A. O'Brien

Abstract

Let $L$ be a finite-dimensional Lie algebra over the field $F$. The Ado-Iwasawa Theorem asserts the existence of a finite-dimensional $L$-module which gives a faithful representation $\rho$ of $L$. Let $S$ be a subnormal subalgebra of $L$, let $\mathcal{F}$ be a saturated formation of soluble Lie algebras and suppose that $S \in \mathcal{F}$. I show that there exists a module $V$ with the extra property that it is $\mathcal{F}$-hypercentral as $S$-module. Further, there exists a module $V$ which has this extra property simultaneously for every such $S$ and $\mathcal{F}$, along with the Hochschild extra that $\rho(x)$ is nilpotent for every $x \in L$ with $\text{ad}(x)$ nilpotent. In particular, if $L$ is supersoluble, then it has a faithful representation by upper triangular matrices.

Keywords and phrases: Lie algebras, faithful representations, saturated formations.

1. Introduction

Let $L$ be a finite-dimensional Lie algebra over the field $F$, which may be of any characteristic. The Ado-Iwasawa Theorem asserts that there exists a faithful finite-dimensional $L$-module $V$. In this paper, I consider some extra properties which we may require of $V$ and of the representation $\rho$ given by $V$. Harish-Chandra [6] and Jacobson [9, Remark, page 203] have proved the characteristic 0 case with the extra property that $\rho(x)$ is nilpotent for all $x$ in the nil radical $N(L)$. Hochschild [7] proved, for any characteristic, that there is a module $V$ with the stronger extra property that $\rho(x)$ is nilpotent for all $x \in L$ for which $\text{ad}(x)$ is nilpotent.

The theory of saturated formations, set out in Barnes and Gastineau-Hills [5] and of $\mathcal{F}$-hypercentral modules, set out in Barnes [1], provides a means of generalising this.
A saturated formation of soluble Lie algebras over $F$ is a class $\mathfrak{F}$ of finite-dimensional soluble Lie algebras over $F$ such that

1. if $L \in \mathfrak{F}$ and $A \triangleleft L$, then $L/A \in \mathfrak{F}$;
2. if $A, B \triangleleft L$ and $L/A, L/B \in \mathfrak{F}$, then $L/(A \cap B) \in \mathfrak{F}$; and
3. if $L/\Phi(L) \in \mathfrak{F}$, then $L \in \mathfrak{F}$,

where $\Phi(L)$ is the Frattini subalgebra of $L$. An irreducible finite-dimensional $L$-module $V$ is called $\mathfrak{F}$-central if the split extension of $V$ by $L/\mathcal{Z}_L(V)$ is in $\mathfrak{F}$, where $\mathcal{Z}_L(V)$ denotes the centraliser of $V$ in $L$. Otherwise, it is called $\mathfrak{F}$-excentric. An $L$-module $V$ is called $\mathfrak{F}$-hypercentral if every composition factor of $V$ is $\mathfrak{F}$-central. It is called $\mathfrak{F}$-hyperexcentric if every composition factor is $\mathfrak{F}$-excentric.

If $S$ is an ideal of $L$, we write $S \triangleleft L$. A subalgebra $S$ of $L$ is called subnormal in $L$, written $S \vartriangleleft L$, if there exists a chain of subalgebras $S = S_0 \triangleleft S_1 \triangleleft \cdots \triangleleft S_r = L$, each an ideal in the next. Let $S$ be a subnormal subalgebra of $L$. Any $L$-module $V$ can be regarded as an $S$-module. To simplify terminology, we say that $V$ is $S\mathfrak{F}$-hypercentral if it is $\mathfrak{F}$-hypercentral as $S$-module and $S\mathfrak{F}$-hyperexcentric if it is $S\mathfrak{F}$-hyperexcentric as $S$-module.

For any field $F$, the class $\mathfrak{N}$ of nilpotent algebras is a saturated formation. If $N$ is a nilpotent Lie algebra, an $N$-module $V$ is $\mathfrak{N}$-hypercentral if and only if every element of $N$ acts nilpotently on $V$. Thus the Harish-Chandra extension of Ado's Theorem asserts, for a finite-dimensional Lie algebra $L$ over a field of characteristic 0, that there exists a faithful, finite-dimensional $L$-module which is $\mathfrak{N}$-hypercentral as $N(L)$-module, where $N(L)$ denotes the nil radical of $L$. We shall generalise this to arbitrary saturated formations $\mathfrak{F}$, with arbitrary subnormal subalgebras $S \in \mathfrak{F}$ in place of $N(L)$. A special case of some interest is that of the saturated formation $\mathfrak{U}$ of supersoluble Lie algebras, that is, of algebras all of whose chief factors are 1-dimensional.

An essential tool for this investigation is the following easy generalisation of Barnes [1, Theorem 4.4].

Lemma 1.1. Let $F$ be any field and let $L$ be a Lie algebra over $F$. Let $\mathfrak{F}$ be a saturated formation of soluble Lie algebras over $F$. Suppose $S \vartriangleleft L$ and that $S \in \mathfrak{F}$. Let $V$ be a finite-dimensional $L$-module. Then $V$ is the $L$-module direct sum $V = V_0 \oplus V_1$, where $V_0$ is $S\mathfrak{F}$-hypercentral and $V_1$ is $S\mathfrak{F}$-hyperexcentric.

Proof. Since $S \vartriangleleft L$, there exists a chain of subalgebras $S = S_0 \triangleleft S_1 \triangleleft \cdots \triangleleft S_r = L$. By Barnes [1, Theorem 4.4], $V$ has an $S$-module direct decomposition $V = V_0 \oplus V_1$ with $V_0$ $S\mathfrak{F}$-hypercentral and $V_1$ $S\mathfrak{F}$-hyperexcentric. We prove by induction over $i$ that $V_0$ and $V_1$ are $S_i$-submodules of $V$.

Let $W$ be any $S_i$-submodule of $V$. For $s \in S_i$, $x \in S_{i+1}$ and $w \in W$, we have $s(xw) = x(sw) + (sx)w \in xW + W$. Thus $xW + W$ is also an $S_i$-submodule of $V$, and $(xW + W)/W$ is a homomorphic image of $W$. If $W$ is $S\mathfrak{F}$-hypercentral, then...
so is \( xW + W \). In particular, for \( W = V_0 \), this implies that \( xV_0 \subseteq V_0 \). Thus \( V_0 \) is invariant under the action of \( S_{i+1} \) and, by induction, under the action of \( L \). Similarly, \( V_1 \) is invariant under the action of \( L \).

Also of use are the following two lemmas proved in Hochschild [7] in the course of proving his main result.

**Lemma 1.2.** Let \( F \) be any field and let \( L \) be a Lie algebra over \( F \) whose derived algebra \( L' \) is nilpotent. Suppose \( x \in L \) and that \( \text{ad}(x) \) is nilpotent. Then \( x \) is in the nilpotent radical \( N(L) \).

**Lemma 1.3.** Suppose \( \text{char}(F) = 0 \). Let \( V \) be a finite-dimensional \( L \)-module giving representation \( \rho \). Suppose \( N(L) \) acts nilpotently on \( V \). Let \( x \in L \) with \( \text{ad}(x) \) nilpotent. Then \( \rho(x) \) is nilpotent.

If \( L \) is a soluble Lie algebra over a field \( F \) of characteristic 0, then \( L' \) is nilpotent. Every subalgebra of a nilpotent Lie algebra is subnormal, so \( x \in N(L) \) implies that the subspace \( \langle x \rangle \) spanned by \( x \) is a subnormal subalgebra of \( L \). Even in non-zero characteristic, the following weak form of Lemma 1.2 holds.

**Lemma 1.4.** Let \( L \) be a soluble Lie algebra over any field \( F \). Suppose \( x \in L \) and that \( \text{ad}(x) \) is nilpotent. Then \( \langle x \rangle \ll L \).

**Proof.** Suppose the result holds for algebras of smaller dimension than \( L \). Let \( A \) be a minimal ideal of \( L \). Then \( A_1 = \langle x \rangle + A \ll L \). But \( A \) is abelian and \( x \) acts nilpotently on \( A \). Thus \( A_1 \) is nilpotent and \( \langle x \rangle \ll A_1 \ll L \).

It follows that, for a module \( V \) giving representation \( \rho \) of a soluble Lie algebra \( L \), the condition that \( \rho(x) \) be nilpotent for all \( x \in L \) with \( \text{ad}(x) \) nilpotent is equivalent to the condition that \( V \) be \( S^2 \)-hypercentral for every nilpotent subnormal subalgebra \( S \) of \( L \).

Suppose \( S \ll L \) and that \( S \in \mathcal{F} \). A straightforward approach to proving the existence of a faithful finite-dimensional \( L \)-module which is \( S^2 \)-hypercentral easily reduces to the case where \( L \) has a unique minimal ideal. We take a faithful finite-dimensional \( L \)-module \( V \). By Lemma 1.1, this is the direct sum of an \( S^2 \)-hypercentral \( L \)-module \( V_0 \) and an \( S^2 \)-hyperexcentric \( L \)-module \( V_1 \). One (at least) of these must be faithful. Unfortunately, it need not be \( V_0 \). That this difficulty is a serious obstruction to the straightforward approach is shown by the results of Section 2.
2. Faithful $\mathfrak{F}$-hyperexcentric modules

To construct faithful $\mathfrak{F}$-hyperexcentric modules, we will use tensor products. The following lemma will help to determine the kernel of a tensor product.

**Lemma 2.1.** Let $L$ be a Lie algebra over any field $F$. Suppose $V$, $W$ are finite-dimensional $L$-modules and that $x$ is in the kernel of $V \otimes W$. Then there exists $\lambda \in F$ such that $xv = \lambda v$ and $xw = -\lambda w$ for all $v \in V$ and $w \in W$.

**Proof.** Let $v$, $w$ be any non-zero elements of $V$ and $W$. Take bases $v = v_0, \ldots, v_m$ and $w = w_0, \ldots, w_n$ of $V$ and $W$. Then $xv = \sum \lambda_i v_i$ and $xw = \sum \mu_j w_j$. Now $0 = x(v \otimes w) = \sum \lambda_i v_i \otimes w_0 + \sum \mu_j v_0 \otimes w_j$. Therefore $\lambda_i = 0$ for $i \neq 0$, $\mu_j = 0$ for $j \neq 0$ and $\lambda_0 + \mu_0 = 0$. Since every non-zero element of $V$ is an eigenvector, $\lambda_0$ is independent of the choice of $v$. □

**Corollary 2.2.** Suppose $x$ is in the kernel of $(W \otimes V) \oplus (W \otimes V \otimes V)$. Then $x$ is in the kernel of $V$.

**Proof.** For $v \in V$ and $w \in W$, we have $xv = \lambda v$ and $xw = -\lambda w$. Then $x(w \otimes v \otimes v) = \lambda(w \otimes v \otimes v)$. Therefore $\lambda = 0$. □

If $\text{char}(F) = 0$, then, by Barnes [2, Theorem 2], for some normal $F$-subspace $\Lambda$ of the algebraic closure $\bar{F}$ of $F$, $\mathfrak{F}$ is the class of all soluble finite-dimensional Lie algebras $S$ over $F$ with the property that for all $x \in S$, the eigenvalues of $\text{ad}(x)$ all lie in $\Lambda$. It follows that, if the degree of $\bar{F}$ over $F$ is finite, there exist Lie algebras $L$ for which the smallest saturated formation $\mathfrak{F}$ containing $L$ is the formation of all soluble Lie algebras.

**Theorem 2.3.** Let $\mathfrak{F}$ be a saturated formation of soluble Lie algebras over the field $F$ of characteristic 0. Suppose $\mathfrak{F}$ is not the formation of all soluble Lie algebras. Let $S \in \mathfrak{F}$ be a non-nilpotent soluble subnormal subalgebra of $L$. Then $L$ has a faithful, finite-dimensional $S\mathfrak{F}$-hyperexcentric module giving representation $\rho$ with $\rho(x)$ nilpotent for all $x \in L$ for which $\text{ad}(x)$ is nilpotent.

**Proof.** Let $N = N(L)$ be the nil radical of $L$. By Lemma 1.3, the condition on an $L$-module $V$ giving representation $\rho$ that $\rho(x)$ be nilpotent for all $x \in L$ with $\text{ad}(x)$ nilpotent is equivalent to $V$ being $N\mathfrak{F}$-hypercentral. By Hochschild [7], $L$ has a faithful finite-dimensional $N\mathfrak{F}$-hypercentral module $V$.

Let $R$ be the soluble radical of $L$. Since $R/N$ is abelian and $S \not\leq N$, there exists a maximal ideal $M \supset N$ of $R$ not containing $S$. Since $LR \leq N$, $M \triangleleft L$. Let $K$ be
the sum of $M$ and a Levy factor of $L$. Then $K$ is an ideal of $L$ of codimension 1 and $K + S = L$.

Let $\mathcal{F}$ be the saturated formation given by the normal subspace $\Lambda$ of $\bar{F}$. Then $\Lambda \neq \mathcal{F}$, so there exists $\alpha \in \bar{F} - \Lambda$. For the 1-dimensional Lie algebra $L/K = \langle \bar{x} \rangle$, we can construct an irreducible module $W$ on which $\bar{x}$ has $\alpha$ as an eigenvalue. Then the $L$-module $W$ is $\mathcal{N}$-hypercentral and $S\mathcal{F}$-excentric.

Let $V_0$ and $V_1$ be the $S\mathcal{F}$-hypercentral and $S\mathcal{F}$-hyperexcentric components of $V$. Put $V^* = (W \otimes V_0) \oplus (W \otimes V_0 \otimes V_0) \oplus V_1$. Then $V^*$ is $\mathcal{N}$-hypercentral and $S\mathcal{F}$-hyperexcentric by Barnes [1, Theorem 2.1] and [4, Theorem 2.3]. If $x$ is in the kernel of $V^*$, then $x$ is in the kernel of $V_1$ and of $(W \otimes V_0) \oplus (W \otimes V_0 \otimes V_0)$. By Corollary 2.2, $x$ is also in the kernel of $V_0$, so $x = 0$. Thus $V^*$ is faithful.

The situation in non-zero characteristic is different. The Lie algebras of nilpotent length at most $n$ form a saturated formation $\mathcal{N}^n$. Thus it is not possible for the smallest saturated formation containing $L$ to be the formation of all soluble Lie algebras. If $L \in \mathcal{N}^n$, then every irreducible $L$-module is $\mathcal{N}^{n+1}$-central. Thus $L$ has no $\mathcal{N}^{n+1}$-hyperexcentric modules. Even when $\mathcal{F}$ is the smallest saturated formation containing the non-nilpotent algebra $L$, there may not be $\mathcal{F}$-hyperexcentric $L$-modules with the Hochschild property. For example, if $L = \langle x, y \rangle$ with $xy = y$ and $F$ is algebraically closed, any irreducible module on which $y$ acts nilpotently is 1-dimensional and so $\mathcal{U}$-central.

**Theorem 2.4.** Suppose $\text{char}(F) \neq 0$. Let $S$ be a soluble subnormal subalgebra of the Lie algebra $L$ over $F$. Let $\mathcal{F}$ be the smallest saturated formation containing $S$. Then $L$ has a faithful finite-dimensional $S\mathcal{F}$-hyperexcentric module.

**Proof.** Let $V$ be a faithful finite-dimensional $L$-module with $V_0$ and $V_1$ its $S\mathcal{F}$-hypercentral and $S\mathcal{F}$-hyperexcentric components. Let $K$ be a minimal ideal of $L$. Let $\mathcal{F}_0$ be the smallest saturated formation containing $(S + K)/K$. If $\mathcal{F}_0 = \mathcal{F}$, then by induction, there exists an irreducible $L/K$-module $W$ which is $(S + K/K)\mathcal{F}$-hyperexcentric. If not, then $S$ is not nilpotent, and since, by Schenkman [10, Theorem 3], $S^\infty \prec L$, we can take $K \subseteq S^\infty$. Since $S \prec L$, the $S$-composition factors of $K$ are isomorphic. As $S \notin \mathcal{F}_0$, $K$ is $S\mathcal{F}_0$-hyperexcentric. Let $\mathcal{F}_1$ be the saturated formation locally defined by $\mathcal{F}_0$, that is, the class of all soluble Lie algebras $M$ with $M/N(M) \in \mathcal{F}_0$. (See [5, Theorem 4.6].) Then $S \in \mathcal{F}_1$. Since by Jacobson [9, Theorem VI.2, page 205], $L$ has a faithful completely reducible module, there exists an irreducible $L$-module $W$ on which $K$ acts faithfully. The $S$-composition factors of $W$ are all isomorphic. Thus $K$ acts non-trivially on each $S$-composition factor $W_i$, $S/\mathcal{G}_S(W_i) \notin \mathcal{F}_0$ and $W$ is $S\mathcal{F}_1$-hyperexcentric. Hence, in either case, we have an irreducible $S\mathcal{F}$-hyperexcentric $L$-module $W$. Put $V^* = (W \otimes V_0) \oplus (W \otimes V_0 \otimes V_0) \oplus V_1$. Then $V^*$ is $S\mathcal{F}$-hyperexcentric. By Corollary 2.2, $V^*$ is faithful. \qed
3. Splitting algebras

To get around the difficulty pointed out above, we follow Iwasawa’s use of a splitting module in the construction of the desired faithful module.

**DEFINITION 3.1.** Let $A$ be an abelian ideal of the Lie algebra $L$. A **splitting algebra** for $L$ relative to $A$ is a Lie algebra $M$ together with an abelian ideal $B$ of $M$ such that $L \leq M$, $L + B = M$, $L \cap B = A$ and such that $M$ splits over $B$.

In the above, we can regard both $A$ and $B$ as $L/A$-modules. Choosing coset representatives in $L$ for the elements of $\tilde{L} = L/A$ by a linear map $u: \tilde{L} \to L$, we can identify $L$ with $\tilde{L} \times A$, identifying $(\tilde{x}, a)$ with the element $u(\tilde{x}) + a \in L$ for $\tilde{x} \in \tilde{L}$ and $a \in A$. We then have the multiplication given by

$$(\tilde{x}_1, a_1)(\tilde{x}_2, a_2) = (\tilde{x}_1\tilde{x}_2, \tilde{x}_1a_2 - \tilde{x}_2a_1 + f(\tilde{x}_1, \tilde{x}_2)),$$

where $f(\tilde{x}_1, \tilde{x}_2) = u(\tilde{x}_1)u(\tilde{x}_2) - u(\tilde{x}_1\tilde{x}_2)$. Then $f: \tilde{L} \times \tilde{L} \to A$ is a 2-cocycle. Let $h$ be the cohomology class of $f$. Let $j^*: H^2(\tilde{L}, A) \to H^2(\tilde{L}, B)$ be the map induced by the module inclusion $j: A \to B$. Then $M$ is the extension of $B$ by $\tilde{L}$ constructed using the cocycle $j^*f$, that is, $M = \tilde{L} \times B$ with multiplication given by

$$(\tilde{x}_1, b_1)(\tilde{x}_2, b_2) = (\tilde{x}_1\tilde{x}_2, \tilde{x}_1b_2 - \tilde{x}_2b_1 + f(\tilde{x}_1, \tilde{x}_2)),$$

for $\tilde{x}_1, \tilde{x}_2 \in \tilde{L}$ and $b_1, b_2 \in B$. The requirement that $M$ splits over $B$ is equivalent to $j^*(h) = 0$.

Since the development of homological algebra, the existence of a splitting algebra has become a triviality. Any $\tilde{L}$-module $A$ has an embedding $j: A \to B$ in an injective module $B$ and we then have $H^2(\tilde{L}, B) = 0$. Except in the trivial case where $\tilde{L} = 0$, the splitting algebra so obtained is infinite-dimensional. The original existence proof constructed the module $B$ from $A$ and the universal enveloping algebra of $\tilde{L}$, also giving an infinite-dimensional splitting algebra. In [8], Iwasawa modified this construction to obtain the following result which was the key to his proof of the Ado-Iwasawa Theorem.

**THEOREM 3.2.** Let $A$ be an abelian ideal of the finite-dimensional Lie algebra $L$ over any field $F$. Then there exists a finite-dimensional splitting algebra for $L$ relative to $A$.

This result can be strengthened in the special case where we have a soluble subnormal subalgebra $S$ of $L$ with $S \in \mathfrak{F}$ for some saturated formation $\mathfrak{F}$ of soluble Lie algebras.
LEMMA 3.3. Let $L$ be a Lie algebra over any field $F$. Suppose $S \ll L$ and that $S \in \mathfrak{F}$ where $\mathfrak{F}$ is a saturated formation of soluble Lie algebras. Let $A$ be an abelian ideal of $L$ which is $S\mathfrak{F}$-hypercentral. Let $h$ be the cohomology class of $L$ as an extension of $A$. Then

1. there exists a finite-dimensional splitting algebra $(M, B)$ for $L$ relative to $A$ with $B$ $S\mathfrak{F}$-hypercentral;
2. there exists an embedding $j : A \rightarrow B$ of $A$ in a finite-dimensional $L/A$-module $B$ which is $S\mathfrak{F}$-hypercentral and such that $j^*(h) = 0$.

PROOF. The two assertions are equivalent. By Iwasawa’s Theorem 3.2, there exists a finite-dimensional splitting algebra $M$ with ideal $B$. For the $L/A$-module inclusion $j : A \rightarrow B$, we have $j^*(h) = 0$. By Lemma 1.1, $B = B_1 \oplus B_1'$ where $B_1$ is $S\mathfrak{F}$-hypercentral and $B_1'$ is $S\mathfrak{F}$-hyperexcentric. As $A$ is $S\mathfrak{F}$-hypercentral, $j(A) \subseteq B_1$ and $j$ is the composite of the inclusion $j_1 : A \rightarrow B_1$ and the inclusion $i_1 : B_1 \rightarrow B$. As the induced map $i_1^*$ of cohomology is injective, it follows that $j_1^*(h) = 0$. Replacing $B$ by $B_1$ gives the result. \hfill \Box

The condition that $A$ be $S\mathfrak{F}$-hypercentral is automatically satisfied if $S \supseteq A$ or if $A$ is central. As the results about splitting algebras will only be needed in the case where $A$ is central, I simplify the statements by assuming this from here on.

We can iterate this reduction of the splitting module. If $(S_2, \mathfrak{F}_2)$ is another pair satisfying the conditions of Lemma 3.3, we can decompose the above module $B_1 = B_2 \oplus B_2'$ where $B_2$ is $S_2\mathfrak{F}_2$-hypercentral and $B_2'$ is $S_2\mathfrak{F}_2$-hyperexcentric. This reduction process must terminate since $B$ is finite-dimensional. We thus have

THEOREM 3.4. Let $A$ be a central ideal of the finite-dimensional Lie algebra $L$ over any field $F$. Then there exists a finite-dimensional splitting algebra $(M, B)$ for $L$ relative to $A$ such that, for every saturated formation $\mathfrak{F}$ and subnormal subalgebra $S \in \mathfrak{F}$, $B$ is $S\mathfrak{F}$-hypercentral.

4. The Hochschild extra

In this section, I show that, if $A$ is central, then there exists a splitting algebra $(M, B)$ as in Theorem 3.4 with the Hochschild extra property that, for all $x \in L$, if $\text{ad}(x)$ is nilpotent, then so is the action $\psi(x)$ of $x$ on $B$. For $N = N(L)$ and the saturated formation $\mathfrak{N}$ of nilpotent algebras, by Theorem 3.4, we may suppose that $B$ is $N\mathfrak{N}$-hypercentral. Thus $\psi(x)$ is nilpotent for all $x \in N$. By Lemma 1.3, we now have
LEMMA 4.1. Let $A$ be a central ideal of the finite-dimensional Lie algebra $L$ over a field of characteristic 0. Then there exists a finite-dimensional splitting algebra $(M, B)$ for $L$ with respect to $A$ which satisfies the extra conditions

1. $B$ is $S\mathcal{F}$-hypercentral for every saturated formation $\mathcal{F}$ and every $S \triangleleft L$ with $S \in \mathcal{F}$;
2. the action $\psi(x)$ of $x$ on $B$ is nilpotent for every $x \in L$ with $\text{ad}(x)$ nilpotent.

Now suppose $\text{char}(F) = p \neq 0$. Then $L$ has a finite-dimensional $p$-envelope $\bar{L}$ by Strade and Farnsteiner [11, Proposition 5.3, page 93]. The $[p]$ operation may be chosen such that $z^{[p]} = 0$ for all $z$ in the centre of $L$. Let $A$ be a central ideal of $L$. Then $A$ is a central $p$-ideal of $\bar{L}$. If $S \triangleleft L$, then $S \triangleleft \bar{L}$. If $B$ is a finite-dimensional $p$-module of $\bar{L}$ which is a splitting module for $\bar{L}$, and so for $L$, with respect to $A$, then it follows as in the proof of Strade and Farnsteiner [11, Theorem 5.4, page 94], that the action $\psi(x)$ of $x$ on $B$ is nilpotent for every $x \in L$ with $\text{ad}(x)$ nilpotent. The following lemma enables us to prove the existence of such a splitting module.

LEMMA 4.2. Let $L$ be a restricted Lie algebra over the field $F$ of characteristic $p$. Let $V$ be an $L$-module of dimension $n$ giving the representation $\rho$. Put $\alpha(x) = \rho(x)^p - \rho(x^{[p]})$. Then $V = V_{[p]} \oplus V_{[p]}'$, where $V_{[p]} = \bigcap_{x \in L} \ker \alpha(x)^n$ is a submodule, all of whose composition factors are $p$-representations, and $V_{[p]}' = \sum_{x \in L} (\alpha(x)^n V$ is a submodule, none of whose composition factors are $p$-representations.

PROOF. Let $x_1, \ldots, x_r$ be a basis of $L$. Put $\tilde{V} = \tilde{F} \otimes_F V$. We take the character decomposition $\tilde{V} = \bigoplus V_i$ corresponding to the characters $S_i$ with $S_0 = 0$. The only eigenvalue of $\alpha(x)$ on $\tilde{V}_i$ is $S_i(x)^p$. If this is non-zero, then $\alpha(x)$ acts invertibly on $\tilde{V}_i$. For all $x \in \bar{L}$, $\alpha(x)^n \tilde{V}_0 = 0$. For each $i > 0$, $S_i \neq 0$ so $S_i(x_j) \neq 0$ for some $x_j$. We thus have

$$\sum_{i > 0} \tilde{V}_i = \sum_{i > 0} \alpha(x_j)^n \tilde{V} = \sum_j \alpha(x_j)^n \tilde{V}.$$ 

It follows that

$$\tilde{V}_0 = \bigcap_{x \in L} \ker \alpha(x)^n = \bigcap_j \ker \alpha(x_j)^n.$$ 

The result follows by linearity. □

THEOREM 4.3. Let $A$ be a central ideal of the finite-dimensional Lie algebra $L$ over any field $F$. Then there exists a finite-dimensional splitting algebra $(M, B)$ for $L$ with respect to $A$ which satisfies the extra conditions

1. $B$ is $S\mathcal{F}$-hypercentral for every saturated formation $\mathcal{F}$ and every $S \triangleleft L$ with $S \in \mathcal{F}$;
the action \( \psi(x) \) of \( x \) on \( B \) is nilpotent for every \( x \in L \) with \( \text{ad}(x) \) nilpotent.

**Proof.** We already have the result if \( \text{char}(F) = 0 \), so suppose \( \text{char}(F) = p \neq 0 \). We embed \( L \) in a finite-dimensional \( p \)-envelope \( \tilde{L} \) with \( z^{[p]} = 0 \) for all \( z \in Z(\tilde{L}) \). By Iwasawa's Theorem 3.2, there exists a finite dimensional splitting module \( B \) for \( \tilde{L} \) relative to \( A \). Since \( A \) is a \( p \)-module, \( A \subseteq B_{[p]} \), and it follows that \( B_{[p]} \) is a splitting module with the property (2). Proceeding as in the proof of Theorem 3.4, we obtain a direct summand of \( B_{[p]} \) which also has the property (1). \( \Box \)

5. The main result

**Theorem 5.1.** Let \( L \) be a finite-dimensional Lie algebra over any field \( F \). Then \( L \) has a faithful finite-dimensional module \( V \) which has the extra properties

1. \( V \) is \( S_\mathcal{F} \)-hypercentral for every saturated formation \( \mathcal{F} \) and every \( S \triangleleft L \) with \( S \in \mathcal{F} \);
2. the action \( \rho(x) \) of \( x \) on \( V \) is nilpotent for every \( x \in L \) with \( \text{ad}(x) \) nilpotent.

**Proof.** The representation of the 1-dimensional algebra by matrices \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) with \( \lambda \in F \) satisfies all the requirements. By induction, we may suppose that the result holds for algebras of smaller dimension than \( \dim(L) \). If \( A_1 \) and \( A_2 \) are distinct minimal ideals of \( L \), then there exist \( L/A_i \)-modules \( V_i \) which satisfy the requirements with respect to \( L/A_i \). The \( L \)-module \( V_1 \oplus V_2 \) then has all the required properties. Thus we may suppose that \( L \) has a unique minimal ideal \( A \).

Since \( L \) is an \( S_\mathcal{F} \)-hypercentral module for every pair \( S \in \mathcal{F} \), \( L/Z \) has a faithful simultaneously \( S_\mathcal{F} \)-hypercentral module, where \( Z \) is the centre of \( L \). Thus the result holds if \( Z = 0 \). Hence we may suppose that \( Z \neq 0 \) and is the unique minimal ideal of \( L \). By Theorem 4.3, there exists a finite-dimensional splitting algebra \( (M, B) \) in which \( B \) and the representation \( \psi \) given by \( B \) have the properties (1) and (2). Let \( L_1 \) be a complement to \( B \) in \( M \). Following Iwasawa, we put \( V = \langle e \rangle \oplus B \) as vector space with action of \( M \) on \( V \) given by \( (x + b)e = b \) and \( (x + b)b' = xb' \), (the product of \( x \) and \( b' \) in \( M \)) for \( x \in L_1 \) and \( b, b' \in B \). Then

\[
(x_1 + b_1)((x_2 + b_2)(\lambda e, b')) = (x_1 + b_1)(0, \lambda b_2 + x_2 b') = (0, \lambda x_1 b_2 + x_1 (x_2 b')).
\]

Denoting the commutator of the actions of \( (x_1 + b_1) \) and \( (x_2 + b_2) \) on \( V \) by \([x_1 + b_1, x_2 + b_2] \), we have

\[
[x_1 + b_1, x_2 + b_2](\lambda e, b') = (0, \lambda x_1 b_2 - \lambda x_2 b_1 + (x_1 x_2) b')
= (x_1 x_2 + x_1 b_2 - x_2 b_1)(\lambda e, b')
= ((x_1 + b_1)(x_2 + b_2))(\lambda e, b').
\]
Thus this action makes $V$ an $M$-module which is clearly finite-dimensional. As $L$ is a subalgebra of $M$, $V$ is an $L$-module. As the unique minimal ideal of $L$ is contained in $B$ which is clearly represented faithfully, $V$ is a faithful $L$-module. $B$ is a submodule of $V$ and is $S_5$-hypercentral while $V/B$ is the trivial module. Thus $V$ is $S_5$-hypercentral for every pair $(S, S)$. As $\rho(x)V \subseteq B$ for all $x \in L$, if $\psi(x)$ is nilpotent on $B$, then $\rho(x)$ is nilpotent on $V$. \hfill \Box

6. $S$-hypercentrality of $p$-modules

Comparison of Lemma 1.1 and Lemma 4.2 suggests a possible link between $p$-modules and $S$-hypercentral modules which would make the non-zero characteristic case of Theorem 5.1 an immediate consequence of Strade and Farnsteiner [11, Theorem 5.4, page 94].

In the following, $F$ is a field of characteristic $p \neq 0$, $F_p$ denotes the field of $p$ elements and $\bar{F}$ the algebraic closure of $F$. A polynomial $f(x)$ over $\bar{F}$ is called $F_p$-linear if the function $f : \bar{F} \to \bar{F}$ given by $f(x)$ is $F_p$-linear. Note that to prove a polynomial $f(x)$ to be $F_p$-linear, it is sufficient to prove $f(a + b) = f(a) + f(b)$ for all $a, b \in \bar{F}$, as then $f(\lambda a) = \lambda f(a)$ for $\lambda \in F_p$ follows. Note also that a polynomial of the form $f(x) = a_0 + a_1x^p + a_2x^{p^2} + \cdots + a_nx^{p^n}$ is $F_p$-linear.

**Lemma 6.1.** If $f(x)$ is $F_p$-linear, then all roots of $f(x)$ have the same multiplicity.

**Proof.** Let $\alpha_1, \ldots, \alpha_n$ be the (not necessarily distinct) roots of $f(x)$. Then $f(x) = a \prod_{i=1}^n(x - \alpha_i)$. For any root $\beta$,

$$f(x) = f(x) + f(\beta) = f(x + \beta) = a \prod_{i=1}^n(x + \beta - \alpha_i).$$

Thus $(x - \alpha_i)$ and $(x + \beta - \alpha_i)$ occur as factors of $f(x)$ with the same multiplicity. But every root $\alpha_j$ is $\alpha_i - \beta$ for some root $\beta$. \hfill \Box

**Lemma 6.2.** Suppose $f(x)$ is $F_p$-linear and that the coefficient of $x$ in $f(x)$ is not zero. Then all roots of $f(x)$ are simple.

**Proof.** Since $f(0) = 0$, there is no constant term. If the roots have multiplicity $r$, then $f(x) = g(x)^r$ and the lowest term of $f(x)$ has degree at least $r$. Hence $r = 1$. \hfill \Box

**Lemma 6.3.** Let $f(x)$ be an $F_p$-linear polynomial. Then $f(x)$ has the form

$$f(x) = a_0x + a_1x^p + a_2x^{p^2} + \cdots + a_nx^{p^n}.$$
PROOF. We use induction over the degree of \( f(x) \). The result holds if the degree is 1. By replacing \( f(x) \) with \( f(x) + x \) if necessary, we may suppose that all roots of \( f(x) \) are simple. The roots of \( f(x) \) form a vector space \( V \) of some finite dimension \( n \) over \( \mathbb{F}_p \). The number of roots is \( p^n \) and as all roots are simple, the degree of \( f(x) \) is \( p^n \). If the leading coefficient is \( a \), then \( g(x) = f(x) - ax^{p^n} \) is \( \mathbb{F}_p \)-linear of lower degree. Therefore \( g(x) \) has the asserted form and the result follows. \( \square \)

**Theorem 6.4.** Let \((L, [p])\) be a restricted Lie algebra over the field \( F \) of characteristic \( p \neq 0 \) and suppose that \( z^{[p]} = 0 \) for all \( z \) in the centre of \( L \). Let \( \mathfrak{z} \) be a saturated formation and suppose \( S \ll L, S \neq 0 \) and \( S \notin \mathfrak{z} \). Let \( V \) be an irreducible \( p \)-module of \( L \). Then \( V \) is \( S\mathfrak{z} \)-hypercentral.

**Proof.** Let \( L, S, V \) be a counterexample with \( L \) of least possible dimension. We now choose \( V \) such that the kernel \( K \) of the representation \( \rho \) of \( L \) on \( V \) has the least possible codimension. Let \( Z = Z(L) \) be the centre of \( L \). Suppose \( Z \neq 0 \). Then \( Z \) acts nilpotently on \( V \) and as \( V \) is irreducible, \( ZV = 0 \). But \( Z \) is a \( p \)-ideal of \( L \), so \( V \) is an irreducible \( p \)-module for the restricted Lie algebra \( L/Z \). As \( V \) is \( (S + Z/Z)\mathfrak{z} \)-hypercentric, \( L/Z \) must have a central element \( \xi \) with \( \xi^{[p]} \neq 0 \), that is, we have \( z \in L \) with \( ad(z)^2 = 0 \) and \( z^{[p]} \notin Z \). Therefore \( Z = 0 \).

If \( A \ll B < L \), then the \( p \)-closure \( A_p \ll B_p \) by Strade and Farnsteiner [11, Proposition 1.3, page 66]. Therefore \( S_p \ll L \). If \( S_p \neq L \), then there exists a \( p \)-ideal \( M \) such that \( S_p \leq M < L \). If \( z \in Z(M) \), then \( ad(z)^2 = 0 \), so \( z^{[p]} \in Z(L) = 0 \). Thus \( M, S \) and any \( M \)-composition factor of \( V \) form a counterexample. Therefore \( S_p = L, L \) is soluble and \( S \ll L \).

Let \( A \) be a minimal ideal of \( S \). Since \( L = S_p, A \ll L \). If \( a \in A \), then \( ad(a)^2 = 0 \), so \( a^{[p]} \in Z \). But \( Z = 0 \). Thus \( A \) is a \( p \)-ideal and \( A V = 0 \) since \( V \) is an irreducible \( p \)-module. There exists an element \( z \) such that \( zL \leq A \), but \( z^{[p]} \notin A \). As \( Z = 0 \), we cannot have \( zA = 0 \), so \( z \) acts invertibly on \( A \). By Barnes [3, Theorem 2.2], \( H^n(L/A; A) = 0 \) for all \( n \) and there exists a subalgebra \( M < L \) which complements \( A \). If \( x \in Z(M) \) and \( xA = 0 \), then \( x \in Z(L) = 0 \). Thus \( Z(M) \cong Z(L/A) \) acts faithfully on \( A \).

There exists a \( p \)-mapping \( [p] \) on \( L/A \) which is zero on \( \tilde{Z} = Z(L/A) \). For any \( \tilde{x} \in \tilde{L} = L/A, \tilde{x}^{[p]} - \tilde{x}^{[p]} \in \tilde{Z} \). Thus any representation of \( \tilde{L} \) whose kernel contains \( \tilde{Z} \) which is a \( p \)-representation with respect to \([p] \) is also a \( p \)-representation with respect to \([p] \). If \( \tilde{Z} \subsetneq K = K/A, \) then \((\tilde{L}, [p]), \tilde{S}, V \) is a counterexample of smaller dimension. Therefore \( \tilde{Z} \subsetneq \tilde{K} \).

Take \( \tilde{z} \in \tilde{Z}, \tilde{z} \notin \tilde{K} \). Since \( \tilde{z} \) is not nilpotent on \( V \), for all \( r, \tilde{z}^{[p]} \notin \tilde{K} \). By replacing \( \tilde{z} \) with \( \tilde{z}^{[p]} \) for some \( r \), we obtain \( \tilde{z} \in \{\tilde{z}^{[p]}, \tilde{z}^{[p]}, \tilde{z}^{[p]}, \ldots\} \). Put \( \tilde{T} = (\tilde{z}, \tilde{z}^{[p]}, \tilde{z}^{[p]}, \ldots). \) Let \( \psi : A \to A \) be the linear transformation of \( A \) given by \( \tilde{z} \).

Let \( r = \dim(\tilde{T}) \). Then there exists a polynomial \( f(x) = x^{p^r} + a_1x^{p^{r-1}} + \cdots + a_rx \)
over $F$ such that $f(\psi) = 0$. Note that the roots of $f(x)$ in the algebraic closure $\bar{F}$ are distinct and form a vector space $\Lambda$ of dimension $r$ over the prime field $\mathbb{F}_p$ of $p$ elements. Let $\Lambda_0$ be the $\mathbb{F}_p$-subspace of $\bar{F}$ spanned by the eigenvalues of $\psi$. Let $m(x)$ be the minimum polynomial of $\psi$ and $\alpha_1, \ldots, \alpha_n$ its roots. Then $\Lambda_0 = (\alpha_1, \ldots, \alpha_n)_{\mathbb{F}_p} \subseteq \Lambda$. Let $s = \dim \Lambda_0$.

Put $g(x) = \prod_{\lambda \in \Lambda_0} (x - \lambda)$. Then $g(x)$ has degree $p^s$. Take any $a \in \bar{F}$ and set $h_a(x) = g(x + a) - g(x) - g(a)$. Since $g(x) = x^{p^s} + \text{terms of lower degree}$, $g(x + a) = (x + a)^{p^s} + \text{lower degree terms} = x^{p^s} + \text{terms of lower degree in } x$ and so $h_a(x)$ is a polynomial of degree less than $p^s$. If $a$ is a root of $g(x)$, then so is $\lambda + a$ for all $\lambda \in \Lambda_0$ and $h_a(\lambda) = 0$. Thus $h_a(x)$ has at least $p^s$ roots and so must be the zero polynomial. Hence $g(x + a) = g(x) + g(a)$ if $g(a) = 0$. Now consider general $a$. For $\lambda \in \Lambda_0$, $g(a + \lambda) = g(a) + g(\lambda)$, so $h_a(\lambda) = g(a + \lambda) - g(\lambda) - g(a) = 0$, so again $h_a(x)$ has at least $p^s$ roots and must be the zero polynomial. Thus $g(x)$ is $\mathbb{F}_p$-linear.

Note also that every automorphism of $\bar{F}$ which fixes $F$ pointwise fixes $g(x)$ which is therefore a polynomial over $F$ since $F(\Lambda)$ is a separable extension of $F$.

Now $f(x)$ is the $\mathbb{F}_p$-linear polynomial over $F$ of least degree for which $f(\psi) = 0$. But $g(\psi) = 0$, so $s \geq r$. But $\Lambda_0$ is an $s$-dimensional subspace of the $r$-dimensional space $\Lambda$. Therefore $\Lambda_0 = \Lambda$.

We now consider the linear transformation $\rho(z) : V \rightarrow V$. Since $\rho$ is a $p$-representation, $f(\rho(z)) = 0$. Thus if $\mu$ is an eigenvalue of $\rho(z)$, then $\mu \in \Lambda = \Lambda_0$. Thus $\mu = \alpha_1 + \cdots + \alpha_k$ for some eigenvalues $\alpha_i$ (not necessarily distinct) of $\psi$. Let $W$ be the $L$-module $\text{Hom}(A^{\otimes k}, V)$ and let $\theta$ be the representation given by $W$. Then $0$ is an eigenvalue of $\theta(z)$.

Since $A$ is $S^3$-hypercentral and $V$ is $S^3$-hyperexcentric, we have by Barnes [1, Theorem 2.1] and [4, Theorem 2.3], that $W$ is $S^3$-hyperexcentric. But for some composition factor $W_0$ of $W$, the action of $z$ on $W_0$ has $0$ as an eigenvalue. Thus $z$ is in the kernel of the representation of $L$ on $W_0$, contrary to the choice of $V$ as giving a representation with kernel of least possible codimension. \hfill $\square$

Any Lie algebra $L$ over a field of characteristic $p$ can be embedded as an ideal in a restricted Lie algebra $(\bar{L}, [p])$ with $z^{[p]} = 0$ for all $z$ in the centre of $\bar{L}$. By Strade and Farnsteiner [11, Theorem 5.4, page 94], $\bar{L}$ has a faithful finite-dimensional $p$-module. As $S \preceq L$ implies $S \preceq \bar{L}$, the characteristic $p$ case of Theorem 5.1 follows by Theorem 6.4.

7. Special cases

We now consider the significance of Theorem 5.1 for supersoluble algebras. A Lie algebra $S$ is supersoluble if it has a sequence $0 = A_0 < A_1 < \cdots < A_n = S$ of ideals of $S$ with $A_i/A_{i-1}$ of dimension $1$ for all $i$. Let $\Omega$ be the saturated formation
of supersoluble algebras. An $S$-module $V$ is $\mathcal{U}$-hypercentral if it has a composition series with all quotients 1-dimensional.

**Theorem 7.1.** Let $L$ be a finite-dimensional Lie algebra over any field $F$ and let $S \ll L$ be supersoluble. Then $L$ has a faithful finite-dimensional representation in which $S$ is represented by upper triangular matrices.

**Proof.** By Theorem 5.1, $L$ has a faithful $S\mathcal{U}$-hypercentral module $V$. It follows that $S$ fixes a flag in $V$ and for suitable choice of basis, is represented by upper triangular matrices.

If $S_i \ll L$ are supersoluble, then by Theorem 5.1, there exists a faithful $L$-module $V$ which is simultaneously $S_i\mathcal{U}$-hypercentral. It does not follow in general that all $S_i$ simultaneously can be represented by upper triangular matrices. Each $S_i$ fixes some flag but there need not be any flag fixed by them all. However this does hold in characteristic 0.

**Lemma 7.2.** Let $L$ be a Lie algebra over a field $F$ of characteristic 0 and let $\mathcal{F}$ be a saturated formation. Let $\{S_i \mid i \in I\}$ be the set of all subnormal subalgebras $S_i \ll L$ which are in $\mathcal{F}$. Put $S = \sum_{i \in I} S_i$. Then $S \ll L$ and $S \in \mathcal{F}$.

**Proof.** Let $R$ be the radical of $L$. Then $LR$ is a nilpotent ideal of $R$. Since $\mathcal{H} \subseteq \mathcal{F}$, $LR \in \mathcal{F}$. Since $S_i$ is soluble and $S_i \ll L$, $S_i \leq R$.

Let $S_1$ be any ideal of $L$ which is in $\mathcal{F}$ and contains $LR$. Let $S_2$ be any subnormal subalgebra of $L$ which is in $\mathcal{F}$. Then $S_1 + S_2 \ll L$. We have to prove $S_1 + S_2 \in \mathcal{F}$. The result then follows.

By Barnes [2, Theorem 2], for some normal $F$-subspace $\Lambda$ of the algebraic closure $\bar{F}$ of $F$, $\mathcal{F}$ is the class of all soluble finite-dimensional Lie algebras $S$ over $F$ with the property that for all $x \in S$, the eigenvalues of $\text{ad}(x)$ all lie in $\Lambda$.

We may suppose $L = S_1 + S_2$. Then $L$ is soluble. Consider any chief factor $V$ of $L$. Then $L'$ is in the kernel of the representation $\rho$ of $L$ on $V$. We have a set $\rho(S_1) \cup \rho(S_2)$ of commuting linear transformations of $V$, all of whose eigenvalues lie in $\Lambda$. They therefore fix a flag in $\bar{F} \otimes V$. For $s_1 \in S_1$ and $s_2 \in S_2$, it follows that the eigenvalues of $\rho(s_1 + s_2)$ are sums of an eigenvalue of $\rho(s_1)$ and an eigenvalue of $s_2$, thus all in $\Lambda$.

**Corollary 7.3.** Let $L$ be a finite-dimensional Lie algebra over a field $F$ of characteristic 0. Then $L$ has a faithful finite-dimensional representation in which every supersoluble subnormal subalgebra of $L$ is represented by upper triangular matrices.
Proof. By Lemma 7.2, there exists a supersoluble ideal $S$ of $L$ which contains every supersoluble subnormal subalgebra. Let $V$ be a faithful $S_{L}$-hypercentral $L$-module. A flag in $V$ fixed by $S$ is fixed by every supersoluble subnormal subalgebra.

Example 7.4. Lemma 7.2 and Corollary 7.3 do not hold in characteristic $p$. Let $V = (v_{0}, \ldots, v_{p-1})$ where the subscripts are integers mod $p$ and let $L = (x, y, z, V)$ with multiplication given by $xy = z, xz = yz = v_{i}v_{j} = 0, xv_{i} = iv_{i-1}, yv_{i} = v_{i+1}$ and $zu_{i} = v_{i}$.

Then $S_{1} = (x, z, V)$ and $S_{2} = (y, z, V)$ are supersoluble ideals of $L$ but $S_{1} + S_{2}$ is not supersoluble. A representation with both $S_{1}$ and $S_{2}$ upper triangular would have $S_{1} + S_{2}$ upper triangular, which would imply $S_{1} + S_{2}$ supersoluble.

Over the field $\mathbb{R}$ of real numbers, there is another saturated formation, $\mathcal{I}$ consisting of those soluble Lie algebras $S$ such that, for all $s \in S$, all eigenvalues of $\text{ad}(s)$ are pure imaginary.

Theorem 7.5. Suppose $S \in \mathcal{I}$ is an ideal of the finite-dimensional Lie algebra $L$ over $\mathbb{R}$. Then $L$ has a faithful finite-dimensional representation in which $S$ is represented by matrices which are block upper triangular, and with the diagonal blocks either 0 or of the form \((0, 0)\) for some $r \in \mathbb{R}$.

Proof. For any soluble Lie algebra $S$ over a field of characteristic 0, the derived subalgebra $S'$ is in the kernel of any irreducible representation. Let $V$ be an $\mathcal{I}$-central irreducible module for $S$ and suppose $s_{1} \in S$ acts non-trivially. Let $s_{2} \in S$. The actions of $s_{1}$ and $s_{2}$ commute, so in the complexification of $V$, they have a common eigenvector. Since the eigenvalues are pure imaginary, for some $r \in \mathbb{R}$, $s_{2} - rs_{1}$ has an eigenvalue 0, thus an eigenvector in $V$. These eigenvectors form a submodule, so by the irreducibility of $V$, $s_{2} - rs_{1}$ acts trivially. It follows that the kernel of the representation has codimension 1 and that $V$ is 2-dimensional with the action of $s_{1}$ given by \((0, 0)\) for some $r \in \mathbb{R}$. The result follows.

References


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