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ADO-IWASAWA EXTRAS

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Abstract

Let L be a finite-dimensional Lie algebra over the field F. The Ado-Iwasawa Theorem asserts the existence of a finite-dimensional L-module which gives a faithful representation ρ of L. Let S be a subnormal subalgebra of L, let $\mathfrak{F}$ be a saturated formation of soluble Lie algebras and suppose that $S \in \mathfrak{F}$. I show that there exists a module V with the extra property that it is $\mathfrak{F}$-hypercentral as S-module. Further, there exists a module V which has this extra property simultaneously for every such S and $\mathfrak{F}$, along with the Hochschild extra that $\rho(x)$ is nilpotent for every $x \in L$ with $\mathrm{ad}(x)$ nilpotent. In particular, if L is supersoluble, then it has a faithful representation by upper triangular matrices.

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1. Introduction

Let L be a finite-dimensional Lie algebra over the field F, which may be of any characteristic. The Ado-Iwasawa Theorem asserts that there exists a faithful finite-dimensional L-module V. In this paper, I consider some extra properties which we may require of V and of the representation ρ given by V. Harish-Chandra [6] and Jacobson [9, Remark, page 203] have proved the characteristic 0 case with the extra property that $\rho(x)$ is nilpotent for all x in the nil radical $N(L)$. Hochschild [7] proved, for any characteristic, that there is a module V with the stronger extra property that $\rho(x)$ is nilpotent for all $x \in L$ for which $\mathrm{ad}(x)$ is nilpotent.

The theory of saturated formations, set out in Barnes and Gastineau-Hills [5] and of $\mathfrak{F}$-hypercentral modules, set out in Barnes [1], provides a means of generalising this.

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A saturated formation of soluble Lie algebras over F is a class $\mathfrak{F}$ of finite-dimensional soluble Lie algebras over F such that

(1) if $L \in \mathfrak{F}$ and $A \lhd L$, then $L/A \in \mathfrak{F}$;
(2) if $A, B \lhd L$ and $L/A, L/B \in \mathfrak{F}$, then $L/(A \cap B) \in \mathfrak{F}$; and
(3) if $L/\Phi(L) \in \mathfrak{F}$, then $L \in \mathfrak{F}$,

where $\Phi(L)$ is the Frattini subalgebra of L. An irreducible finite-dimensional L-module V is called $\mathfrak{F}$-central if the split extension of V by $L/\mathscr{C}_L(V)$ is in $\mathfrak{F}$, where $\mathscr{C}_L(V)$ denotes the centraliser of V in L. Otherwise, it is called $\mathfrak{F}$-excentric. An L-module V is called $\mathfrak{F}$-hypercentral if every composition factor of V is $\mathfrak{F}$-central. It is called $\mathfrak{F}$-hyperexcentric if every composition factor is $\mathfrak{F}$-excentric.

If S is an ideal of L, we write $S \lhd L$. A subalgebra S of L is called subnormal in L, written $S \lhd\lhd L$, if there exists a chain of subalgebras $S = S_0 \lhd S_1 \lhd \cdots \lhd S_r = L$, each an ideal in the next. Let S be a subnormal subalgebra of L. Any L-module V can be regarded as an S-module. To simplify terminology, we say that V is $S\mathfrak{F}$-hypercentral if it is $\mathfrak{F}$-hypercentral as S-module and $S\mathfrak{F}$-hyperexcentric if it is $\mathfrak{F}$-hyperexcentric as S-module.

For any field F, the class $\mathfrak{N}$ of nilpotent algebras is a saturated formation. If N is a nilpotent Lie algebra, an N-module V is $\mathfrak{N}$-hypercentral if and only if every element of N acts nilpotently on V. Thus the Harish-Chandra extension of Ado's Theorem asserts, for a finite-dimensional Lie algebra L over a field of characteristic 0, that there exists a faithful, finite-dimensional L-module which is $\mathfrak{N}$-hypercentral as $N(L)$-module, where $N(L)$ denotes the nil radical of L. We shall generalise this to arbitrary saturated formations $\mathfrak{F}$, with arbitrary subnormal subalgebras $S \in \mathfrak{F}$ in place of $N(L)$. A special case of some interest is that of the saturated formation $\mathfrak{U}$ of supersoluble Lie algebras, that is, of algebras all of whose chief factors are 1-dimensional.

An essential tool for this investigation is the following easy generalisation of Barnes [1, Theorem 4.4].

LEMMA 1.1. *Let F be any field and let L be a Lie algebra over F. Let $\mathfrak{F}$ be a saturated formation of soluble Lie algebras over F. Suppose $S \lhd\lhd L$ and that $S \in \mathfrak{F}$. Let V be a finite-dimensional L-module. Then V is the L-module direct sum $V = V_0 \oplus V_1$, where V_0 is $S\mathfrak{F}$-hypercentral and V_1 is $S\mathfrak{F}$-hyperexcentric.*

PROOF. Since $S \lhd\lhd L$, there exists a chain of subalgebras $S = S_0 \lhd S_1 \lhd \cdots \lhd S_r = L$. By Barnes [1, Theorem 4.4], V has an S-module direct decomposition $V = V_0 \oplus V_1$ with V_0 $\mathfrak{F}$-hypercentral and V_1 $\mathfrak{F}$-hyperexcentric. We prove by induction over i that V_0 and V_1 are S_i-submodules of V.

Let W be any S_i-submodule of V. For $s \in S_i$, $x \in S_{i+1}$ and $w \in W$, we have $s(xw) = x(sw) + (sx)w \in xW + W$. Thus $xW + W$ is also an S_i-submodule of V, and $(xW + W)/W$ is a homomorphic image of W. If W is $S\mathfrak{F}$-hypercentral, then so is $xW + W$. In particular, for $W = V_0$, this implies that $xV_0 \subseteq V_0$. Thus V_0 is invariant under the action of S_{i+1} and, by induction, under the action of L. Similarly, V_1 is invariant under the action of L. □

Also of use are the following two lemmas proved in Hochschild [7] in the course of proving his main result.

LEMMA 1.2. *Let F be any field and let L be a Lie algebra over F whose derived algebra L' is nilpotent. Suppose $x \in L$ and that* $\mathrm{ad}(x)$ *is nilpotent. Then x is in the nilpotent radical $N(L)$.*

LEMMA 1.3. *Suppose* $\mathrm{char}(F) = 0$. *Let V be a finite-dimensional L-module giving representation ρ. Suppose $N(L)$ acts nilpotently on V. Let $x \in L$ with* $\mathrm{ad}(x)$ *nilpotent. Then $\rho(x)$ is nilpotent.*

If L is a soluble Lie algebra over a field F of characteristic 0, then L' is nilpotent. Every subalgebra of a nilpotent Lie algebra is subnormal, so $x \in N(L)$ implies that the subspace $\langle x \rangle$ spanned by x is a subnormal subalgebra of L. Even in non-zero characteristic, the following weak form of Lemma 1.2 holds.

LEMMA 1.4. *Let L be a soluble Lie algebra over any field F. Suppose $x \in L$ and that* $\mathrm{ad}(x)$ *is nilpotent. Then $\langle x \rangle \lhd\lhd L$.*

PROOF. Suppose the result holds for algebras of smaller dimension than L. Let A be a minimal ideal of L. Then $A_1 = \langle x \rangle + A \lhd\lhd L$. But A is abelian and x acts nilpotently on A. Thus A_1 is nilpotent and $\langle x \rangle \lhd\lhd A_1 \lhd\lhd L$. □

It follows that, for a module V giving representation ρ of a soluble Lie algebra L, the condition that $\rho(x)$ be nilpotent for all $x \in L$ with $\mathrm{ad}(x)$ nilpotent is equivalent to the condition that V be $S\mathfrak{N}$-hypercentral for every nilpotent subnormal subalgebra S of L.

Suppose $S \lhd\lhd L$ and that $S \in \mathfrak{F}$. A straightforward approach to proving the existence of a faithful finite-dimensional L-module which is $S\mathfrak{F}$-hypercentral easily reduces to the case where L has a unique minimal ideal. We take a faithful finite-dimensional L-module V. By Lemma 1.1, this is the direct sum of an $S\mathfrak{F}$-hypercentral L-module V_0 and an $S\mathfrak{F}$-hyperexcentric L-module V_1. One (at least) of these must be faithful. Unfortunately, it need not be V_0. That this difficulty is a serious obstruction to the straightforward approach is shown by the results of Section 2.

2. Faithful $\mathfrak{F}$-hyperexcentric modules

To construct faithful $\mathfrak{F}$-hyperexcentric modules, we will use tensor products. The following lemma will help to determine the kernel of a tensor product.

LEMMA 2.1. *Let L be a Lie algebra over any field F. Suppose V, W are finite-dimensional L-modules and that x is in the kernel of $V \otimes W$. Then there exists $\lambda \in F$ such that $xv = \lambda v$ and $xw = -\lambda w$ for all $v \in V$ and $w \in W$.*

PROOF. Let v, w be any non-zero elements of V and W. Take bases $v = v_0, \dots, v_m$ and $w = w_0, \dots, w_n$ of V and W. Then $xv = \sum \lambda_i v_i$ and $xw = \sum \mu_j w_j$. Now $0 = x(v \otimes w) = \sum \lambda_i v_i \otimes w_0 + \sum \mu_j v_0 \otimes w_j$. Therefore $\lambda_i = 0$ for $i \neq 0$, $\mu_j = 0$ for $j \neq 0$ and $\lambda_0 + \mu_0 = 0$. Since every non-zero element of V is an eigenvector, λ_0 is independent of the choice of v. □

COROLLARY 2.2. *Suppose x is in the kernel of $(W \otimes V) \oplus (W \otimes V \otimes V)$. Then x is in the kernel of V.*

PROOF. For $v \in V$ and $w \in W$, we have $xv = \lambda v$ and $xw = -\lambda w$. Then $x(w \otimes v \otimes v) = \lambda(w \otimes v \otimes v)$. Therefore $\lambda = 0$. □

If $\mathrm{char}(F) = 0$, then, by Barnes [2, Theorem 2], for some normal F-subspace Λ of the algebraic closure $\bar{F}$ of F, $\mathfrak{F}$ is the class of all soluble finite-dimensional Lie algebras S over F with the property that for all $x \in S$, the eigenvalues of $\mathrm{ad}(x)$ all lie in Λ. It follows that, if the degree of $\bar{F}$ over F is finite, there exist Lie algebras L for which the smallest saturated formation $\mathfrak{F}$ containing L is the formation of all soluble Lie algebras.

THEOREM 2.3. *Let $\mathfrak{F}$ be a saturated formation of soluble Lie algebras over the field F of characteristic 0. Suppose $\mathfrak{F}$ is not the formation of all soluble Lie algebras. Let $S \in \mathfrak{F}$ be a non-nilpotent soluble subnormal subalgebra of L. Then L has a faithful, finite-dimensional $S\mathfrak{F}$-hyperexcentric module giving representation ρ with $\rho(x)$ nilpotent for all $x \in L$ for which $\mathrm{ad}(x)$ is nilpotent.*

PROOF. Let $N = N(L)$ be the nil radical of L. By Lemma 1.3, the condition on an L-module V giving representation ρ that $\rho(x)$ be nilpotent for all $x \in L$ with $\mathrm{ad}(x)$ nilpotent is equivalent to V being $N\mathfrak{N}$-hypercentral. By Hochschild [7], L has a faithful finite-dimensional $N\mathfrak{N}$-hypercentral module V.

Let R be the soluble radical of L. Since R/N is abelian and $S \not\leq N$, there exists a maximal ideal $M \geq N$ of R not containing S. Since $LR \leq N$, $M \triangleleft L$. Let K be the sum of M and a Levy factor of L. Then K is an ideal of L of codimension 1 and $K + S = L$.

Let $\mathfrak{F}$ be the saturated formation given by the normal subspace Λ of $\bar{F}$. Then $\Lambda \neq \bar{F}$, so there exists $\alpha \in \bar{F} - \Lambda$. For the 1-dimensional Lie algebra $L/K = \langle \bar{x} \rangle$, we can construct an irreducible module W on which $\bar{x}$ has α as an eigenvalue. Then the L-module W is $N\mathfrak{N}$-hypercentral and $S\mathfrak{F}$-excentric.

Let V_0 and V_1 be the $S\mathfrak{F}$-hypercentral and $S\mathfrak{F}$-hyperexcentric components of V. Put $V^* = (W \otimes V_0) \oplus (W \otimes V_0 \otimes V_0) \oplus V_1$. Then V^* is $N\mathfrak{N}$-hypercentral and $S\mathfrak{F}$-hyperexcentric by Barnes [1, Theorem 2.1] and [4, Theorem 2.3]. If x is in the kernel of V^*, then x is in the kernel of V_1 and of $(W \otimes V_0) \oplus (W \otimes V_0 \otimes V_0)$. By Corollary 2.2, x is also in the kernel of V_0, so $x = 0$. Thus V^* is faithful. □

The situation in non-zero characteristic is different. The Lie algebras of nilpotent length at most n form a saturated formation $\mathfrak{N}^n$. Thus it is not possible for the smallest saturated formation containing L to be the formation of all soluble Lie algebras. If $L \in \mathfrak{N}^n$, then every irreducible L-module is $\mathfrak{N}^{n+1}$-central. Thus L has no $\mathfrak{N}^{n+1}$-hyperexcentric modules. Even when $\mathfrak{F}$ is the smallest saturated formation containing the non-nilpotent algebra L, there may not be $\mathfrak{F}$-hyperexcentric L-modules with the Hochschild property. For example, if $L = \langle x, y \rangle$ with $xy = y$ and F is algebraically closed, any irreducible module on which y acts nilpotently is 1-dimensional and so $\mathfrak{U}$-central.

THEOREM 2.4. *Suppose* char$(F) \neq 0$. *Let S be a soluble subnormal subalgebra of the Lie algebra L over F. Let $\mathfrak{F}$ be the smallest saturated formation containing S. Then L has a faithful finite-dimensional $S\mathfrak{F}$-hyperexcentric module.*

PROOF. Let V be a faithful finite-dimensional L-module with V_0 and V_1 its $S\mathfrak{F}$-hypercentral and $S\mathfrak{F}$-hyperexcentric components. Let K be a minimal ideal of L. Let $\mathfrak{F}_0$ be the smallest saturated formation containing $(S + K)/K$. If $\mathfrak{F}_0 = \mathfrak{F}$, then by induction, there exists an irreducible L/K-module W which is $(S + K/K)\mathfrak{F}$-hyperexcentric. If not, then S is not nilpotent, and since, by Schenkman [10, Theorem 3], $S^\infty \lhd L$, we can take $K \subseteq S^\infty$. Since $S \lhd\lhd L$, the S-composition factors of K are isomorphic. As $S \notin \mathfrak{F}_0$, K is $S\mathfrak{F}_0$-hyperexcentric. Let $\mathfrak{F}_1$ be the saturated formation locally defined by $\mathfrak{F}_0$, that is, the class of all soluble Lie algebras M with $M/N(M) \in \mathfrak{F}_0$. (See [5, Theorem 4.6].) Then $S \in \mathfrak{F}_1$. Since by Jacobson [9, Theorem VI.2, page 205], L has a faithful completely reducible module, there exists an irreducible L-module W on which K acts faithfully. The S-composition factors of W are all isomorphic. Thus K acts non-trivially on each S-composition factor W_i, $S/\mathscr{C}_S(W_i) \notin \mathfrak{F}_0$ and W is $S\mathfrak{F}_1$-hyperexcentric. Hence, in either case, we have an irreducible $S\mathfrak{F}$-hyperexcentric L-module W. Put $V^* = (W \otimes V_0) \oplus (W \otimes V_0 \otimes V_0) \oplus V_1$. Then V^* is $S\mathfrak{F}$-hyperexcentric. By Corollary 2.2, V^* is faithful. □

3. Splitting algebras

To get around the difficulty pointed out above, we follow Iwasawa's use of a splitting module in the construction of the desired faithful module.

DEFINITION 3.1. Let A be an abelian ideal of the Lie algebra L. A *splitting algebra* for L relative to A is a Lie algebra M together with an abelian ideal B of M such that $L \leq M$, $L + B = M$, $L \cap B = A$ and such that M splits over B.

In the above, we can regard both A and B as L/A-modules. Choosing coset representatives in L for the elements of $\bar{L} = L/A$ by a linear map $u : \bar{L} \to L$, we can identify L with $\bar{L} \times A$, identifying $(\bar{x}, a)$ with the element $u(\bar{x}) + a \in L$ for $\bar{x} \in \bar{L}$ and $a \in A$. We then have the multiplication given by

$$(\bar{x}_1, a_1)(\bar{x}_2, a_2) = \big(\bar{x}_1\bar{x}_2, \bar{x}_1 a_2 - \bar{x}_2 a_1 + f(\bar{x}_1, \bar{x}_2)\big),$$

where $f(\bar{x}_1, \bar{x}_2) = u(\bar{x}_1)u(\bar{x}_2) - u(\bar{x}_1\bar{x}_2)$. Then $f : \bar{L} \wedge \bar{L} \to A$ is a 2-cocycle. Let h be the cohomology class of f. Let $j^* : H^2(\bar{L}, A) \to H^2(\bar{L}, B)$ be the map induced by the module inclusion $j : A \to B$. Then M is the extension of B by $\bar{L}$ constructed using the cocycle jf, that is, $M = \bar{L} \times B$ with multiplication given by

$$(\bar{x}_1, b_1)(\bar{x}_2, b_2) = \big(\bar{x}_1\bar{x}_2, \bar{x}_1 b_2 - \bar{x}_2 b_1 + f(\bar{x}_1, \bar{x}_2)\big),$$

for $\bar{x}_1, \bar{x}_2 \in \bar{L}$ and $b_1, b_2 \in B$. The requirement that M splits over B is equivalent to $j^*(h) = 0$.

Since the development of homological algebra, the existence of a splitting algebra has become a triviality. Any $\bar{L}$-module A has an embedding $j : A \to B$ in an injective module B and we then have $H^2(\bar{L}, B) = 0$. Except in the trivial case where $\bar{L} = 0$, the splitting algebra so obtained is infinite-dimensional. The original existence proof constructed the module B from A and the universal enveloping algebra of $\bar{L}$, also giving an infinite-dimensional splitting algebra. In [8], Iwasawa modified this construction to obtain the following result which was the key to his proof of the Ado-Iwasawa Theorem.

THEOREM 3.2. *Let A be an abelian ideal of the finite-dimensional Lie algebra L over any field F. Then there exists a finite-dimensional splitting algebra for L relative to A.*

This result can be strengthened in the special case where we have a soluble subnormal subalgebra S of L with $S \in \mathfrak{F}$ for some saturated formation $\mathfrak{F}$ of soluble Lie algebras.

LEMMA 3.3. *Let L be a Lie algebra over any field F. Suppose $S \lhd\lhd L$ and that $S \in \mathfrak{F}$ where $\mathfrak{F}$ is a saturated formation of soluble Lie algebras. Let A be an abelian ideal of L which is $S\mathfrak{F}$-hypercentral. Let h be the cohomology class of L as an extension of A. Then*

(1) *there exists a finite-dimensional splitting algebra (M, B) for L relative to A with B $S\mathfrak{F}$-hypercentral;*
(2) *there exists an embedding $j : A \to B$ of A in a finite-dimensional L/A-module B which is $S\mathfrak{F}$-hypercentral and such that $j^*(h) = 0$.*

PROOF. The two assertions are equivalent. By Iwasawa's Theorem 3.2, there exists a finite-dimensional splitting algebra M with ideal B. For the L/A-module inclusion $j : A \to B$, we have $j^*(h) = 0$. By Lemma 1.1, $B = B_1 \oplus B_1'$ where B_1 is $S\mathfrak{F}$-hypercentral and B_1' is $S\mathfrak{F}$-hyperexcentric. As A is $S\mathfrak{F}$-hypercentral, $j(A) \subseteq B_1$ and j is the composite of the inclusion $j_1 : A \to B_1$ and the inclusion $i_1 : B_1 \to B$. As the induced map i_1^* of cohomology is injective, it follows that $j_1^*(h) = 0$. Replacing B by B_1 gives the result. □

The condition that A be $S\mathfrak{F}$-hypercentral is automatically satisfied if $S \supseteq A$ or if A is central. As the results about splitting algebras will only be needed in the case where A is central, I simplify the statements by assuming this from here on.

We can iterate this reduction of the splitting module. If $(S_2, \mathfrak{F}_2)$ is another pair satisfying the conditions of Lemma 3.3, we can decompose the above module $B_1 = B_2 \oplus B_2'$ where B_2 is $S_2\mathfrak{F}_2$-hypercentral and B_2' is $S_2\mathfrak{F}_2$-hyperexcentric. This reduction process must terminate since B is finite-dimensional. We thus have

THEOREM 3.4. *Let A be a central ideal of the finite-dimensional Lie algebra L over any field F. Then there exists a finite-dimensional splitting algebra (M, B) for L relative to A such that, for every saturated formation $\mathfrak{F}$ and subnormal subalgebra $S \in \mathfrak{F}$, B is $S\mathfrak{F}$-hypercentral.*

4. The Hochschild extra

In this section, I show that, if A is central, then there exists a splitting algebra (M, B) as in Theorem 3.4 with the Hochschild extra property that, for all $x \in L$, if $\mathrm{ad}(x)$ is nilpotent, then so is the action $\psi(x)$ of x on B. For $N = N(L)$ and the saturated formation $\mathfrak{N}$ of nilpotent algebras, by Theorem 3.4, we may suppose that B is $N\mathfrak{N}$-hypercentral. Thus $\psi(x)$ is nilpotent for all $x \in N$. By Lemma 1.3, we now have

LEMMA 4.1. *Let A be a central ideal of the finite-dimensional Lie algebra L over a field of characteristic 0. Then there exists a finite-dimensional splitting algebra (M, B) for L with respect to A which satisfies the extra conditions*

(1) *B is $S\mathfrak{F}$-hypercentral for every saturated formation $\mathfrak{F}$ and every $S \lhd\lhd L$ with $S \in \mathfrak{F}$;*
(2) *the action $\psi(x)$ of x on B is nilpotent for every $x \in L$ with* $\mathrm{ad}(x)$ *nilpotent.*

Now suppose $\mathrm{char}(F) = p \neq 0$. Then L has a finite-dimensional p-envelope $\bar{L}$ by Strade and Farnsteiner [11, Proposition 5.3, page 93]. The $[p]$ operation may be chosen such that $z^{[p]} = 0$ for all z in the centre of $\bar{L}$. Let A be a central ideal of L. Then A is a central p-ideal of $\bar{L}$. If $S \lhd\lhd L$, then $S \lhd\lhd \bar{L}$. If B is a finite-dimensional p-module of $\bar{L}$ which is a splitting module for $\bar{L}$, and so for L, with respect to A, then it follows as in the proof of Strade and Farnsteiner [11, Theorem 5.4, page 94], that the action $\psi(x)$ of x on B is nilpotent for every $x \in L$ with $\mathrm{ad}(x)$ nilpotent. The following lemma enables us to prove the existence of such a splitting module.

LEMMA 4.2. *Let L be a restricted Lie algebra over the field F of characteristic p. Let V be an L-module of dimension n giving the representation ρ. Put $\alpha(x) = \rho(x)^p - \rho(x^{[p]})$. Then $V = V_{[p]} \oplus V_{[p']}$, where $V_{[p]} = \bigcap_{x \in L} \ker \alpha(x)^n$ is a submodule, all of whose composition factors are p-representations, and $V_{[p']} = \sum_{x\in L} \alpha(x)^n V$ is a submodule, none of whose composition factors are p-representations.*

PROOF. Let $x_1, \dots, x_r$ be a basis of L. Put $\bar{V} = \bar{F} \otimes_F V$. We take the character decomposition $\bar{V} = \sum_i \bar{V}_i$ corresponding to the characters S_i with $S_0 = 0$. The only eigenvalue of $\alpha(x)$ on $\bar{V}_i$ is $S_i(x)^p$. If this is non-zero, then $\alpha(x)$ acts invertibly on $\bar{V}_i$. For all $x \in \bar{L}$, $\alpha(x)^n \bar{V}_0 = 0$. For each $i > 0$, $S_i \neq 0$ so $S_i(x_{j_i}) \neq 0$ for some x_{j_i}. We thus have

$$\sum_{i>0} \bar{V}_i = \sum_{i>0} \alpha(x_{j_i})^n \bar{V} = \sum_j \alpha(x_j)^n \bar{V}.$$

It follows that

$$\bar{V}_0 = \bigcap_{x\in\bar{L}} \ker \alpha(x)^n = \bigcap_j \ker \alpha(x_j)^n.$$

The result follows by linearity. $\square$

THEOREM 4.3. *Let A be a central ideal of the finite-dimensional Lie algebra L over any field F. Then there exists a finite-dimensional splitting algebra (M, B) for L with respect to A which satisfies the extra conditions*

(1) *B is $S\mathfrak{F}$-hypercentral for every saturated formation $\mathfrak{F}$ and every $S \lhd\lhd L$ with $S \in \mathfrak{F}$;*

(2) *the action $\psi(x)$ of x on B is nilpotent for every $x \in L$ with* $\mathrm{ad}(x)$ *nilpotent.*

PROOF. We already have the result if $\mathrm{char}(F) = 0$, so suppose $\mathrm{char}(F) = p \neq 0$. We embed L in a finite-dimensional p-envelope $\bar{L}$ with $z^{[p]} = 0$ for all $z \in Z(\bar{L})$. By Iwasawa's Theorem 3.2, there exists a finite dimensional splitting module B for $\bar{L}$ relative to A. Since A is a p-module, $A \subseteq B_{[p]}$, and it follows that $B_{[p]}$ is a splitting module with the property (2). Proceeding as in the proof of Theorem 3.4, we obtain a direct summand of $B_{[p]}$ which also has the property (1). □

5. The main result

THEOREM 5.1. *Let L be a finite-dimensional Lie algebra over any field F. Then L has a faithful finite-dimensional module V which has the extra properties*

(1) *V is $S\mathfrak{F}$-hypercentral for every saturated formation $\mathfrak{F}$ and every $S \lhd\lhd L$ with $S \in \mathfrak{F}$;*

(2) *the action $\rho(x)$ of x on V is nilpotent for every $x \in L$ with* $\mathrm{ad}(x)$ *nilpotent.*

PROOF. The representation of the 1-dimensional algebra by matrices $\left(\begin{smallmatrix} 0 & 0 \\ \lambda & 0 \end{smallmatrix}\right)$ with $\lambda \in F$ satisfies all the requirements. By induction, we may suppose that the result holds for algebras of smaller dimension than $\dim(L)$. If A_1 and A_2 are distinct minimal ideals of L, then there exist L/A_i-modules V_i which satisfy the requirements with respect to L/A_i. The L-module $V_1 \oplus V_2$ then has all the required properties. Thus we may suppose that L has a unique minimal ideal A.

Since L is an $S\mathfrak{F}$-hypercentral module for every pair $S \in \mathfrak{F}$, L/Z has a faithful simultaneously $S\mathfrak{F}$-hypercentral module, where Z is the centre of L. Thus the result holds if $Z = 0$. Hence we may suppose that $Z \neq 0$ and is the unique minimal ideal of L. By Theorem 4.3, there exists a finite-dimensional splitting algebra (M, B) in which B and the representation ψ given by B have the properties (1) and (2). Let L_1 be a complement to B in M. Following Iwasawa, we put $V = \langle e \rangle \oplus B$ as vector space with action of M on V given by $(x + b)e = b$ and $(x + b)b' = xb'$, (the product of x and b' in M) for $x \in L_1$ and $b, b' \in B$. Then

$$(x_1 + b_1)\big((x_2 + b_2)(\lambda e, b')\big) = (x_1 + b_1)(0, \lambda b_2 + x_2 b') = \big(0, \lambda x_1 b_2 + x_1(x_2 b')\big).$$

Denoting the commutator of the actions of $(x_1 + b_1)$ and $(x_2 + b_2)$ on V by $[x_1 + b_1, x_2 + b_2]$, we have

$$\begin{aligned}
[x_1 + b_1, x_2 + b_2](\lambda e, b') &= \big(0, \lambda x_1 b_2 - \lambda x_2 b_1 + (x_1 x_2)b'\big) \\
&= (x_1 x_2 + x_1 b_2 - x_2 b_1)(\lambda e, b') \\
&= \big((x_1 + b_1)(x_2 + b_2)\big)(\lambda e, b').
\end{aligned}$$

Thus this action makes V an M-module which is clearly finite-dimensional. As L is a subalgebra of M, V is an L-module. As the unique minimal ideal of L is contained in B which is clearly represented faithfully, V is a faithful L-module. B is a submodule of V and is $S\mathfrak{F}$-hypercentral while V/B is the trivial module. Thus V is $S\mathfrak{F}$-hypercentral for every pair $(S, \mathfrak{F})$. As $\rho(x)V \subseteq B$ for all $x \in L$, if $\psi(x)$ is nilpotent on B, then $\rho(x)$ is nilpotent on V. □

6. $\mathfrak{F}$-hypercentrality of p-modules

Comparison of Lemma 1.1 and Lemma 4.2 suggests a possible link between p-modules and $\mathfrak{F}$-hypercentral modules which would make the non-zero characteristic case of Theorem 5.1 an immediate consequence of Strade and Farnsteiner [11, Theorem 5.4, page 94].

In the following, F is a field of characteristic $p \neq 0$, $\mathbb{F}_p$ denotes the field of p elements and $\bar{F}$ the algebraic closure of F. A polynomial $f(x)$ over $\bar{F}$ is called $\mathbb{F}_p$-linear if the function $f : \bar{F} \to \bar{F}$ given by $f(x)$ is $\mathbb{F}_p$-linear. Note that to prove a polynomial $f(x)$ to be $\mathbb{F}_p$-linear, it is sufficient to prove $f(a+b) = f(a) + f(b)$ for all $a, b \in \bar{F}$, as then $f\lambda a) = \lambda f(a)$ for $\lambda \in \mathbb{F}_p$ follows. Note also that a polynomial of the form $f(x) = a_0x + a_1x^p + a_2x^{p^2} + \cdots + a_nx^{p^n}$ is $\mathbb{F}_p$-linear.

LEMMA 6.1. *If $f(x)$ is $\mathbb{F}_p$-linear, then all roots of $f(x)$ have the same multiplicity.*

PROOF. Let $\alpha_1, \dots, \alpha_n$ be the (not necessarily distinct) roots of $f(x)$. Then $f(x) = a\prod_{i=1}^n (x - \alpha_i)$. For any root β,

$$f(x) = f(x) + f(\beta) = f(x+\beta) = a\prod_{i=1}^{n}(x + \beta - \alpha_i).$$

Thus $(x - \alpha_i)$ and $(x + \beta - \alpha_i)$ occur as factors of $f(x)$ with the same multiplicity. But every root α_j is $\alpha_i - \beta$ for some root β. □

LEMMA 6.2. *Suppose $f(x)$ is $\mathbb{F}_p$-linear and that the coefficient of x in $f(x)$ is not zero. Then all roots of $f(x)$ are simple.*

PROOF. Since $f(0) = 0$, there is no constant term. If the roots have multiplicity r, then $f(x) = g(x)^r$ and the lowest term of $f(x)$ has degree at least r. Hence $r = 1$. □

LEMMA 6.3. *Let $f(x)$ be an $\mathbb{F}_p$-linear polynomial. Then $f(x)$ has the form*

$$f(x) = a_0x + a_1x^p + a_2x^{p^2} + \cdots + a_nx^{p^n}.$$

PROOF. We use induction over the degree of $f(x)$. The result holds if the degree is 1. By replacing $f(x)$ with $f(x)+x$ if necessary, we may suppose that all roots of $f(x)$ are simple. The roots of $f(x)$ form a vector space V of some finite dimension n over $\mathbb{F}_p$. The number of roots is p^n and as all roots are simple, the degree of $f(x)$ is p^n. If the leading coefficient is a, then $g(x) = f(x) - ax^{p^n}$ is $\mathbb{F}_p$-linear of lower degree. Therefore $g(x)$ has the asserted form and the result follows. □

THEOREM 6.4. *Let $(L, [p])$ be a restricted Lie algebra over the field F of characteristic $p \neq 0$ and suppose that $z^{[p]} = 0$ for all z in the centre of L. Let $\mathfrak{F}$ be a saturated formation and suppose $S \lhd\lhd L$, $S \neq 0$ and $S \in \mathfrak{F}$. Let V be an irreducible p-module of L. Then V is $S\mathfrak{F}$-hypercentral.*

PROOF. Let L, S, V be a counterexample with L of least possible dimension. We now choose V such that the kernel K of the representation ρ of L on V has the least possible codimension. Let $Z = Z(L)$ be the centre of L. Suppose $Z \neq 0$. Then Z acts nilpotently on V and as V is irreducible, $ZV = 0$. But Z is a p-ideal of L, so V is an irreducible p-module for the restricted Lie algebra L/Z. As V is $(S + Z/Z)\mathfrak{F}$-hyperexcentric, L/Z must have a central element $\bar{z}$ with $\bar{z}^{[p]} \neq 0$, that is, we have $z \in L$ with $\mathrm{ad}(z)^2 = 0$ and $z^{[p]} \notin Z$. Therefore $Z = 0$.

If $A \lhd B < L$, then the p-closure $A_p \lhd B_p$ by Strade and Farnsteiner [11, Proposition 1.3, page 66]. Therefore $S_p \lhd\lhd L$. If $S_p \neq L$, then there exists a p-ideal M such that $S_p \leq M < L$. If $z \in Z(M)$, then $\mathrm{ad}(z)^2 = 0$, so $z^{[p]} \in Z(L) = 0$. Thus M, S and any M-composition factor of V form a counterexample. Therefore $S_p = L$, L is soluble and $S \lhd L$.

Let A be a minimal ideal of S. Since $L = S_p$, $A \lhd L$. If $a \in A$, then $\mathrm{ad}(a)^2 = 0$, so $a^{[p]} \in Z$. But $Z = 0$. Thus A is a p-ideal and $AV = 0$ since V is an irreducible p-module. There exists an element z such that $zL \leq A$, but $z^{[p]} \notin A$. As $Z = 0$, we cannot have $zA = 0$, so z acts invertibly on A. By Barnes [3, Theorem 2.2], $H^n(L/A; A) = 0$ for all n and there exists a subalgebra $M < L$ which complements A. If $x \in Z(M)$ and $xA = 0$, then $x \in Z(L) = 0$. Thus $Z(M) \simeq Z(L/A)$ acts faithfully on A.

There exists a p-mapping $[p]'$ on L/A which is zero on $\bar{Z} = Z(L/A)$. For any $\bar{x} \in \bar{L} = L/A$, $\bar{x}^{[p]} - \bar{x}^{[p]'} \in \bar{Z}$. Thus any representation of $\bar{L}$ whose kernel contains $\bar{Z}$ which is a p-representation with respect to $[p]$ is also a p-representation with respect to $[p]'$. If $\bar{Z} \subseteq \bar{K} = K/A$, then $(\bar{L}, [p]')$, $\bar{S}$, V is a counterexample of smaller dimension. Therefore $\bar{Z} \not\subseteq \bar{K}$.

Take $\bar{z} \in \bar{Z}$, $\bar{z} \notin \bar{K}$. Since $\bar{z}$ is not nilpotent on V, for all r, $\bar{z}^{[p]^r} \notin \bar{K}$. By replacing $\bar{z}$ with $\bar{z}^{[p]^r}$ for some r, we obtain $\bar{z} \in \langle \bar{z}^{[p]}, \bar{z}^{[p]^2}, \bar{z}^{[p]^3}, \dots \rangle$. Put $\bar{T} = \langle \bar{z}, \bar{z}^{[p]}, \bar{z}^{[p]^2}, \dots \rangle$. Let $\psi : A \to A$ be the linear transformation of A given by $\bar{z}$.

Let $r = \dim(\bar{T})$. Then there exists a polynomial $f(x) = x^{p^r} + a_1 x^{p^{r-1}} + \dots + a_r x$

over F such that $f(\psi) = 0$. Note that the roots of $f(x)$ in the algebraic closure $\bar{F}$ are distinct and form a vector space Λ of dimension r over the prime field $\mathbb{F}_p$ of p elements. Let Λ_0 be the $\mathbb{F}_p$-subspace of $\bar{F}$ spanned by the eigenvalues of ψ. Let $m(x)$ be the minimum polynomial of ψ and $\alpha_1, \dots, \alpha_n$ its roots. Then $\Lambda_0 = \langle \alpha_1, \dots, \alpha_n \rangle_{\mathbb{F}_p} \subseteq \Lambda$. Let $s = \dim \Lambda_0$.

Put $g(x) = \Pi_{\lambda \in \Lambda_0}(x - \lambda)$. Then $g(x)$ has degree p^s. Take any $a \in \bar{F}$ and set $h_a(x) = g(x+a) - g(x) - g(a)$. Since $g(x) = x^{p^s}$ + terms of lower degree, $g(x+a) = (x+a)^{p^s}$ + lower degree terms $= x^{p^s}$ + terms of lower degree in x and so $h_a(x)$ is a polynomial of degree less than p^s. If a is a root of $g(x)$, then so is $\lambda + a$ for all $\lambda \in \Lambda_0$ and $h_a(\lambda) = 0$. Thus $h_a(x)$ has at least p^s roots and so must be the zero polynomial. Hence $g(x+a) = g(x) + g(a)$ if $g(a) = 0$. Now consider general a. For $\lambda \in \Lambda_0$, $g(a+\lambda) = g(a) + g(\lambda)$, so $h_a(\lambda) = g(a+\lambda) - g(\lambda) - g(a) = 0$, so again $h_a(x)$ has at least p^s roots and must be the zero polynomial. Thus $g(x)$ is $\mathbb{F}_p$-linear. Note also that every automorphism of $\bar{F}$ which fixes F pointwise fixes $g(x)$ which is therefore a polynomial over F since $F(\Lambda)$ is a separable extension of F.

Now $f(x)$ is the $\mathbb{F}_p$-linear polynomial over F of least degree for which $f(\psi) = 0$. But $g(\psi) = 0$, so $s \geq r$. But Λ_0 is an s-dimensional subspace of the r-dimensional space Λ. Therefore $\Lambda_0 = \Lambda$.

We now consider the linear transformation $\rho(z) : V \to V$. Since ρ is a p-representation, $f(\rho(z)) = 0$. Thus if μ is an eigenvalue of $\rho(z)$, then $\mu \in \Lambda = \Lambda_0$. Thus $\mu = \alpha_1 + \dots + \alpha_k$ for some eigenvalues α_i (not necessarily distinct) of ψ. Let W be the L-module $\mathrm{Hom}(A^{\otimes k}, V)$ and let θ be the representation given by W. Then 0 is an eigenvalue of $\theta(z)$.

Since A is $S\mathfrak{F}$-hypercentral and V is $S\mathfrak{F}$-hyperexcentric, we have by Barnes [1, Theorem 2.1] and [4, Theorem 2.3], that W is $S\mathfrak{F}$-hyperexcentric. But for some composition factor W_0 of W, the action of z on W_0 has 0 as an eigenvalue. Thus z is in the kernel of the representation of L on W_0, contrary to the choice of V as giving a representation with kernel of least possible codimension. $\square$

Any Lie algebra L over a field of characteristic p can be embedded as an ideal in a restricted Lie algebra $(\bar{L}, [p])$ with $z^{[p]} = 0$ for all z in the centre of $\bar{L}$. By Strade and Farnsteiner [11, Theorem 5.4, page 94], $\bar{L}$ has a faithful finite-dimensional p-module. As $S \lhd\lhd L$ implies $S \lhd\lhd \bar{L}$, the characteristic p case of Theorem 5.1 follows by Theorem 6.4.

7. Special cases

We now consider the significance of Theorem 5.1 for supersoluble algebras. A Lie algebra S is supersoluble if it has a sequence $0 = A_0 < A_1 < \dots < A_n = S$ of ideals of S with A_i/A_{i-1} of dimension 1 for all i. Let $\mathfrak{U}$ be the saturated formation of supersoluble algebras. An S-module V is $\mathfrak{U}$-hypercentral if it has a composition series with all quotients 1-dimensional.

THEOREM 7.1. *Let L be a finite-dimensional Lie algebra over any field F and let $S \lhd\lhd L$ be supersoluble. Then L has a faithful finite-dimensional representation in which S is represented by upper triangular matrices.*

PROOF. By Theorem 5.1, L has a faithful $S\mathfrak{U}$-hypercentral module V. It follows that S fixes a flag in V and for suitable choice of basis, is represented by upper triangular matrices. □

If $S_i \lhd\lhd L$ are supersoluble, then by Theorem 5.1, there exists a faithful L-module V which is simultaneously $S_i\mathfrak{U}$-hypercentral. It does not follow in general that all S_i simultaneously can be represented by upper triangular matrices. Each S_i fixes some flag but there need not be any flag fixed by them all. However this does hold in characteristic 0.

LEMMA 7.2. *Let L be a Lie algebra over a field F of characteristic 0 and let $\mathfrak{F}$ be a saturated formation. Let $\{S_i \mid i \in I\}$ be the set of all subnormal subalgebras $S_i \lhd\lhd L$ which are in $\mathfrak{F}$. Put $S = \sum_{i \in I} S_i$. Then $S \lhd L$ and $S \in \mathfrak{F}$.*

PROOF. Let R be the radical of L. Then LR is a nilpotent ideal of R. Since $\mathfrak{N} \subseteq \mathfrak{F}$, $LR \in \mathfrak{F}$. Since S_i is soluble and $S_i \lhd\lhd L$, $S_i \leq R$.

Let S_1 be any ideal of L which is in $\mathfrak{F}$ and contains LR. Let S_2 be any subnormal subalgebra of L which is in $\mathfrak{F}$. Then $S_1 + S_2 \lhd L$. We have to prove $S_1 + S_2 \in \mathfrak{F}$. The result then follows.

By Barnes [2, Theorem 2], for some normal F-subspace Λ of the algebraic closure $\bar{F}$ of F, $\mathfrak{F}$ is the class of all soluble finite-dimensional Lie algebras S over F with the property that for all $x \in S$, the eigenvalues of $\mathrm{ad}(x)$ all lie in Λ.

We may suppose $L = S_1 + S_2$. Then L is soluble. Consider any chief factor V of L. Then L' is in the kernel of the representation ρ of L on V. We have a set $\rho(S_1) \cup \rho(S_2)$ of commuting linear transformations of V, all of whose eigenvalues lie in Λ. They therefore fix a flag in $\bar{F} \otimes V$. For $s_1 \in S_1$ and $s_2 \in S_2$, it follows that the eigenvalues of $\rho(s_1 + s_2)$ are sums of an eigenvalue of $\rho(s_1)$ and an eigenvalue of s_2, thus all in Λ. □

COROLLARY 7.3. *Let L be a finite-dimensional Lie algebra over a field F of characteristic 0. Then L has a faithful finite-dimensional representation in which every supersoluble subnormal subalgebra of L is represented by upper triangular matrices.*

PROOF. By Lemma 7.2, there exists a supersoluble ideal S of L which contains every supersoluble subnormal subalgebra. Let V be a faithful $S\mathfrak{U}$-hypercentral L-module. A flag in V fixed by S is fixed by every supersoluble subnormal subalgebra. □

EXAMPLE 7.4. Lemma 7.2 and Corollary 7.3 do not hold in characteristic p. Let $V = \langle v_0, \dots, v_{p-1}\rangle$ where the subscripts are integers mod p and let $L = \langle x, y, z, V\rangle$ with multiplication given by $xy = z$, $xz = yz = v_i v_j = 0$, $xv_i = iv_{i-1}$, $yv_i = v_{i+1}$ and $zv_i = v_i$. Then $S_1 = \langle x, z, V\rangle$ and $S_2 = \langle y, z, V\rangle$ are supersoluble ideals of L but $S_1 + S_2$ is not supersoluble. A representation with both S_1 and S_2 upper triangular would have $S_1 + S_2$ upper triangular, which would imply $S_1 + S_2$ supersoluble.

Over the field $\mathbb{R}$ of real numbers, there is another saturated formation, $\mathfrak{I}$ consisting of those soluble Lie algebras S such that, for all $s \in S$, all eigenvalues of $\operatorname{ad}(s)$ are pure imaginary.

THEOREM 7.5. *Suppose $S \in \mathfrak{I}$ is an ideal of the finite-dimensional Lie algebra L over $\mathbb{R}$. Then L has a faithful finite-dimensional representation in which S is represented by matrices which are block upper triangular, and with the diagonal blocks either* 0 *or of the form* $\left(\begin{smallmatrix} 0 & r \\ -r & 0 \end{smallmatrix}\right)$ *for some $r \in \mathbb{R}$.*

PROOF. For any soluble Lie algebra S over a field of characteristic 0, the derived subalgebra S' is in the kernel of any irreducible representation. Let V be an $\mathfrak{I}$-central irreducible module for S and suppose $s_1 \in S$ acts non-trivially. Let $s_2 \in S$. The actions of s_1 and s_2 commute, so in the complexification of V, they have a common eigenvector. Since the eigenvalues are pure imaginary, for some $r \in \mathbb{R}$, $s_2 - rs_1$ has an eigenvalue 0, thus an eigenvector in V. These eigenvectors form a submodule, so by the irreducibility of V, $s_2 - rs_1$ acts trivially. It follows that the kernel of the representation has codimension 1 and that V is 2-dimensional with the action of s_1 given by $\left(\begin{smallmatrix} 0 & r \\ -r & 0 \end{smallmatrix}\right)$ for some $r \in \mathbb{R}$. The result follows. □

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